Added mass and damping of structures with angular symmetry

by S. Malenica[1] & J. N. Newman[2]

[1] Bureau Veritas, France <sime.malenica@bureauveritas.com>

[2] MIT and WAMIT, Inc., USA <jnn@mit.edu>

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1 Introduction

Structures that are axisymmetric about a vertical axis have obvious symmetry properties. With respect to the added-mass and damping coefficients, these are unchanged by rotation of the body or coordinate system about the vertical axis, through an arbitrary angle θ. In such cases the coefficients for surge and sway are identical, with no coupling between these modes. The same properties apply for roll and pitch. Our objective here is to show that similar properties exist for structures that are not axisymmetric, if the geometry is unchanged by rotation about the vertical axis through an angle $\theta = 2\pi/N$ with N an integer and $N \geq 3$. The examples include structures with multiple columns or floats that are equally spaced around a circle, and single cylinders with polygonal shape such as equilateral triangles or pentagons. Figure 1 shows several examples with $N = 3$.

The symmetry properties to be derived here are obvious for cases such as a square cylinder or square array of circular columns, but not for other cases, especially if N is odd. Unlike circular and square cylinders, the radiated waves can be quite different for surge and sway, as shown for the triangular cylinder in Figure 2. Thus it is surprising to find that the added mass and damping matrices for this structure have almost the same forms as for axisymmetric bodies; the only difference is a nonzero moment due to yaw. Two alternative proofs are presented in the following Sections.

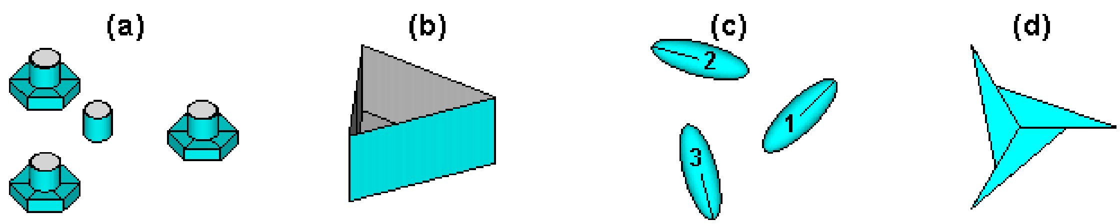

Figure 1: Examples of structures where the geometry is unchanged by rotation about the vertical axis through the angle $2\pi/3$. (a): wind-turbine floats; (b) equilateral triangular cylinder; (c) spheroids at 45° angles; (d) 6-sided cylinder. (a) and (b) are symmetrical about the vertical planes that include the center of the structure and a body or vertex; (c) and (d) are asymmetric.

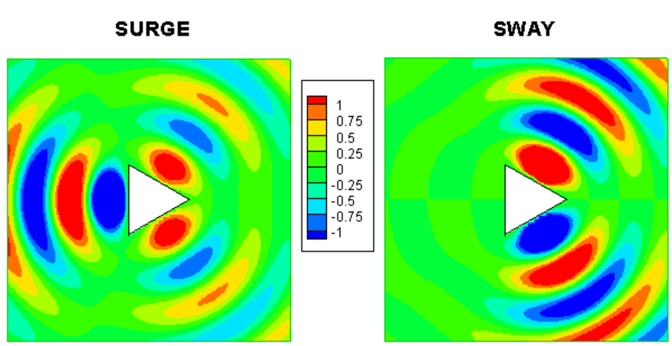

Figure 2: Contour plots of the free-surface elevations due to surge and sway of the triangular cylinder shown in Figure 1(b). The cylinder sides are 2m wide by 1m draft, the fluid depth is infinite and the wavelength in the far field is 2m. The elevations shown in the legend are per unit amplitude of surge and sway.

2 Symmetry relations based on rotation of the coordinate system

The structure is assumed to be rigid, with six degrees of motion. The added mass and damping are represented by 6×6 matrices $\mathbf{A}$ and $\mathbf{B}$, with coefficients A_{ij} and B_{ij}. The row index i represents the three components of the force ($i = 1, 2, 3$) and moment ($i = 4, 5, 6$). The column index j represents the modes of translation ($j = 1, 2, 3$) and rotation ($j = 4, 5, 6$). These are defined with respect to the Cartesian coordinate system x, y, z, with the z-axis vertical. $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices, with $A_{ij} = A_{ji}$ and $B_{ij} = B_{ji}$. Only the added mass is considered in the following analysis; the derivation for the damping matrix is identical.

If the matrix $\mathbf{A}$ is partitioned into four 3×3 sub-matrices $\mathbf{A}_{mn}$, and the coordinate system is rotated about the z-axis by the angle θ, the matrix $\mathbf{A}^*$ in the new system is given by

$$\mathbf{A}^* = \begin{bmatrix} \mathbf{A}^*{}_{11} & \mathbf{A}^*{}_{12} \\ \mathbf{A}^*{}_{21} & \mathbf{A}^*{}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}\mathbf{A}_{11}\mathbf{Q}^T & \mathbf{Q}\mathbf{A}_{12}\mathbf{Q}^T \\ \mathbf{Q}\mathbf{A}_{21}\mathbf{Q}^T & \mathbf{Q}\mathbf{A}_{22}\mathbf{Q}^T \end{bmatrix} \tag{1}$$

where the transformation matrix $\mathbf{Q}$ is defined in the form

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Here $c = \cos\theta$ and $s = \sin\theta$. After evaluating the matrix products the sub-matrix $\mathbf{A}^*{}_{11}$ is

$$\mathbf{A}^*_{11} = \begin{bmatrix} A_{11}c^2 + A_{12}sc + A_{21}sc + A_{22}s^2 & -A_{11}sc + A_{12}c^2 - A_{21}s^2 + A_{22}sc & A_{13}c + A_{23}s \\ -A_{11}cs - A_{12}s^2 + A_{21}c^2 + A_{22}cs & A_{11}s^2 - A_{12}cs - A_{21}sc + A_{22}c^2 & -A_{13}s + A_{23}c \\ A_{31}c + A_{32}s & -A_{31}s + A_{32}c & A_{33} \end{bmatrix}. \tag{3}$$

Since the geometry of the structure is unchanged by rotation through an angle $\theta = 2\pi/N$, rotation of the coordinates through the same angle does not change the added-mass matrix. Thus $\mathbf{A}^*{}_{11} = \mathbf{A}_{11}$. Equating the coefficients in these two sub-matrices gives a set of equations which can be reduced to the following forms if $s \neq 0$:

$$\left.\begin{aligned} (A_{11} - A_{22})s - (A_{12} + A_{21})c = 0 \\ (A_{11} - A_{22})c + (A_{12} + A_{21})s = 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} A_{13}(1-c) - A_{23}s = 0 \\ A_{13}s + A_{23}(1-c) = 0 \end{aligned}\right\}. \tag{4}$$

These equations are homogeneous and the determinants $s^2 + c^2$ and $s^2 + (1-c)^2$ are nonzero. Thus

$$A_{11} - A_{22} = 0, \quad A_{12} + A_{21} = 0, \quad A_{13} = 0, \quad A_{23} = 0. \tag{5}$$

Following the same procedure for the other sub-matrices gives the results

$$A_{14} - A_{25} = 0, \quad A_{15} + A_{24} = 0, \quad A_{16} = 0, \quad A_{26} = 0, \tag{6}$$

$$A_{41} - A_{52} = 0, \quad A_{42} + A_{51} = 0, \quad A_{43} = 0, \quad A_{53} = 0, \tag{7}$$

$$A_{44} - A_{55} = 0, \quad A_{45} + A_{54} = 0, \quad A_{46} = 0, \quad A_{56} = 0. \tag{8}$$

Since $\mathbf{A}$ is symmetric, $A_{12} = A_{21} = 0$ and $A_{45} = A_{54} = 0$. After using (5-8) and imposing symmetry it follows that

$$\mathbf{A} = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} & A_{15} & 0 \\ 0 & A_{11} & 0 & -A_{15} & A_{14} & 0 \\ 0 & 0 & A_{33} & 0 & 0 & A_{36} \\ A_{14} & -A_{15} & 0 & A_{44} & 0 & 0 \\ A_{15} & A_{14} & 0 & 0 & A_{44} & 0 \\ 0 & 0 & A_{36} & 0 & 0 & A_{66} \end{bmatrix}. \tag{9}$$

These results are based on the fact that $\mathbf{A}^* = \mathbf{A}$ when $\theta = 2\pi/N$, but they imply more general conclusions. Indeed, they have been derived without explicitly assigning the angle θ of the rotated coordinate system. Since the equations (4) are homogeneous the solutions (5) do not depend on θ, and similarly for the solutions (6-8) for the other sub-matrices. Thus the matrix $\mathbf{A}^*$ is independent of θ. This means that the added mass and damping are independent of the angle of rotation of the coordinate system, as in the case of an axisymmetric structure.

If the structure is symmetrical about the $x-z$ plane, as in Figure 1(a,b), $A_{14} = 0$ and $A_{36} = 0$. In that case the only difference in (9) relative to an axisymmetric structure is the nonzero coefficient A_{66}, representing the added moment of inertia due to yaw.

3 Symmetry relations based on the individual force matrix for one body

A triangular array with $N = 3$ bodies is considered to simplify the analysis. The bodies are centered on a circle at polar angles $\theta_n = (n-1)(2\pi/3)$ $(n = 1, 2, 3)$ or, equivalently, at $\theta = 0$ and $\theta = \pm(2\pi/3)$. The entire structure moves as a rigid body with six degrees of freedom, as in Section 2. The added mass corresponding to the force and moment on the body n is represented by the 6×6 matrix $\mathbf{a}^{(n)}$ with coefficients $a_{ij}^{(n)}$. In general these matrices are full and asymmetric, as in the example described in Section 4. The matrix for the entire structure is

$$\mathbf{A} = \mathbf{a}^{(1)} + \mathbf{a}^{(2)} + \mathbf{a}^{(3)}. \tag{10}$$

Since the geometry is unchanged by rotation through the angles $\theta = \pm(2\pi/3)$ it follows that $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$ can be related to $\mathbf{a}^{(1)}$ using the transformation matrix (2) with 3×3 sub-matrices, in a similar manner to (1) except that the transformation here is in the opposite sense. Thus for $(n = 2, 3)$,

$$\mathbf{a}^{(n)} = \begin{bmatrix} \mathbf{a}_{11}^{(n)} & \mathbf{a}_{12}^{(n)} \\ \mathbf{a}_{21}^{(n)} & \mathbf{a}_{22}^{(n)} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_n^T \mathbf{a}_{11}^{(1)} \mathbf{Q}_n & \mathbf{Q}_n^T \mathbf{a}_{12}^{(1)} \mathbf{Q}_n \\ \mathbf{Q}_n^T \mathbf{a}_{21}^{(1)} \mathbf{Q}_n & \mathbf{Q}_n^T \mathbf{a}_{22}^{(1)} \mathbf{Q}_n \end{bmatrix} \tag{11}$$

where $\mathbf{Q}_n$ is defined by (2) with $\theta = \theta_n$.

In the following equations it is convenient to omit the superscript 1 for the coefficients of the matrix $\mathbf{a}^{(1)}$. Thus $a_{ij} \equiv a_{ij}^{(1)}$. After the indicated multiplications the results are similar to (3) except for the signs of terms which are linear in $s = \sin\theta_n$. However these terms cancel when the sum in (9) is evaluated, since $\sin\theta_3 = -\sin\theta_2$, and it follows that

$$\mathbf{A}_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + 2 \begin{bmatrix} a_{11}c^2 + a_{22}s^2 & a_{12}c^2 - a_{21}s^2 & a_{13}c \\ -a_{12}s^2 + a_{21}c^2 & a_{11}s^2 + a_{22}c^2 & a_{23}c \\ a_{31}c & a_{32}c & a_{33} \end{bmatrix}. \tag{12}$$

After substituting $c = \cos(2\pi/3) = -1/2$, $c^2 = 1/4$ and $s^2 = 3/4$, and combining the two matrices in (12),

$$\mathbf{A}_{11} = (3/2) \begin{bmatrix} a_{11} + a_{22} & a_{12} - a_{21} & 0 \\ a_{21} - a_{12} & a_{11} + a_{22} & 0 \\ 0 & 0 & 2a_{33} \end{bmatrix}. \tag{13}$$

Thus

$$A_{11} = A_{22} = \tfrac{3}{2}(a_{11} + a_{22}), \tag{14}$$

$$A_{33} = 3a_{33}. \tag{15}$$

Since $\mathbf{A}$ is symmetric, $a_{12} - a_{21} = 0$ and it follows that

$$A_{12} = A_{21} = 0. \tag{16}$$

Repeating the same process for the other sub-matrices gives the results

$$A_{44} = A_{55} = \tfrac{3}{2}(a_{44} + a_{55}), \tag{17}$$

$$A_{66} = 3a_{66}, \tag{18}$$

$$A_{14} = A_{25} = A_{41} = A_{52} = \tfrac{3}{2}(a_{14} + a_{25}) = \tfrac{3}{2}(a_{41} + a_{52}), \tag{19}$$

$$A_{15} = -A_{24} = A_{51} = -A_{42} = \tfrac{3}{2}(a_{15} - a_{24}) = \tfrac{3}{2}(a_{51} - a_{42}), \tag{20}$$

$$A_{36} = A_{63} = 3a_{36} = 3a_{63}. \tag{21}$$

All of the other coefficients A_{ij} not included in (14-21) are equal to zero. These results are consistent with (9).

This analysis has been described for three separate bodies, but it can be applied more generally to structures such as those shown in Figures 1(a,b,d) simply by dividing the submerged surface into three angular sectors and replacing the pressure force on each body by the force acting on the corresponding sector. The extension for $N > 3$ follows by summing (10) over all bodies. If N is odd the same analysis can be followed for each pair of bodies which are symmetrically centered about $\theta = 0$. The final results are unchanged, except that the factors 3 and $\frac{3}{2}$ in (14-21) are replaced by N and $N/2$.

4 Numerical example

Table 1 shows the added-mass and damping coefficients for the spheroids in Figure 1(c). The spheroids are prolate, with length 3m and diameter 1m. The centers are on a circle of radius 2m. The axis of each spheroid is in the plane of the free surface, oriented at 45° from the tangent to the circle. The added-mass coefficients are normalized by the fluid density ρ and the damping coefficients by $\rho\omega$ where ω is the frequency. The wavenumber $K = \omega^2/g = 1m^{-1}$ and the fluid depth is infinite.

The matrices **A** and **B** in the upper part of Table 1 are the coefficients for the complete structure. These have the same form as the matrix (9). Since this structure is asymmetrical A_{14}, A_{25} and A_{36} are nonzero and $A_{14} = A_{25}$. The same properties apply for B_{14}, B_{25} and B_{36}. (If the spheroids were oriented at zero or 90° relative to the tangent the structure would be symmetric about the $x - z$ plane and these coefficients would all be equal to zero.) The matrices $\mathbf{a}^{(1)}$ and $\mathbf{b}^{(1)}$ in the lower part of Table 1 represent the force and moment on spheroid 1. The coefficients for spheroids 2 and 3 can be evaluated using (11); the sums for all three spheroids are the same as the coefficients in **A** and **B**.

A

1.3164	0.0000	0.0000	-0.0036	-0.3448	0.0000
0.0000	1.3164	0.0000	0.3448	-0.0036	0.0000
0.0000	0.0000	0.2601	0.0000	0.0000	1.0643
-0.0036	0.3448	0.0000	5.2618	0.0000	0.0000
-0.3448	-0.0036	0.0000	0.0000	5.2618	0.0000
0.0000	0.0000	1.0643	0.0000	0.0000	6.9598

B

0.3122	0.0000	0.0000	0.0967	0.5644	0.0000
0.0000	0.3122	0.0000	-0.5644	0.0967	0.0000
0.0000	0.0000	0.6887	0.0000	0.0000	-0.2616
0.0967	-0.5644	0.0000	3.9414	0.0000	0.0000
0.5644	0.0967	0.0000	0.0000	3.9414	0.0000
0.0000	0.0000	-0.2616	0.0000	0.0000	1.9662

$\mathbf{a}^{(1)}$

0.4482	-0.3717	-0.4244	-0.1956	-0.1112	-0.6891
-0.3717	0.4294	0.1799	0.1187	0.1932	1.0465
0.1657	-0.0055	0.0867	0.0034	-1.6749	0.3548
-0.0575	-0.0441	0.1674	0.0757	-0.0825	-0.2543
-0.2739	0.0551	-0.3408	-0.0825	3.4322	-0.4552
-0.7727	0.8578	0.3548	0.2253	0.4006	2.3199

$\mathbf{b}^{(1)}$

0.0438	-0.1141	0.1751	0.0026	0.1497	-0.4613
-0.1141	0.1643	-0.0817	-0.2266	0.0618	0.3423
-0.1946	0.0089	0.2296	-0.0032	-1.1651	-0.0872
0.0476	-0.0346	-0.0639	0.1519	-0.1454	0.0824
0.3417	0.0169	-0.3952	-0.1454	2.4756	0.0920
-0.2156	0.3451	-0.0872	-0.4331	0.1678	0.6554

Table 1: Added-mass (left) and damping coefficients (right) for the spheroids shown in Figure 1(c). The upper matrices **A** and **B** are for the complete structure. The lower matrices $\mathbf{a}^{(1)}$ and $\mathbf{b}^{(1)}$ are for the individual force and moment on the first spheroid.

5 Discussion

Structures with angular symmetry have been considered, where the geometry is unchanged by rotation about the vertical axis through an angle $2\pi/N$, with N an integer and $N \geq 3$. For this type of structure the added-mass and damping coefficients for surge and sway are equal, and uncoupled. The same properties apply for roll and pitch. The general form of the coefficients is shown in the matrix (9). If the structure is symmetric about a vertical plane, as in most cases of practical importance, the only difference compared to an axisymmetric structure is the nonzero added moment of inertia due to yaw.

These properties apply only to the force and moment due to body motions, and not to the exciting forces in incident waves or other properties such as the radiated wave patterns shown in Figure 2. However there are integral relations which apply. For example the total energy in the radiated waves is related to the damping coefficients; thus the two wave patterns shown in Figure 2 radiate the same amount of energy. Similarly, the integrals of the square of the exciting forces over all wave directions can be related to the damping coefficients using the Haskind relations; thus the integrals of the squares of the exciting forces in surge and sway (or the moments in roll and pitch) have the same value.

The case $N = 2$ is an exception. When $\theta = \pi$ the coefficients of $\mathbf{A}^*$ are equal to $\pm A_{ij}$, providing no relations between different added-mass or damping coefficients. A vertical flat plate in the $x - z$ plane is an obvious example, where the surge added mass is zero and the sway added mass is nonzero.