Generalised hybrid element method for oscillatory motion of multiple floating structures in oblique regular waves.

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HIGHLIGHTS

- The hydrodynamics and wave interaction with multiple surface piercing floating structures with arbitrary cross-sections are examined.
- The system is modelled under linear water wave theory and the boundary problem is solved using a *hybrid element method* combining eigenfunction expansion and boundary element method.
- The simulation involves instantaneous coupling the equations of motion for the moultiple floating structures and the fluid potentials.
- The hydrodynamics are studied using frequency dependent coefficients for wave forces, wave reflection and transmission.

1. INTRODUCTION

The hydrodynamics of floating structures are essential in design and engineering of the categories ocean structures like wave energy devices, free surface breakwaters, offshore platforms, ships etc., which has a significant impact in the areas of coastal engineering, environmental protection, recreation, and various marine facility designs. A significant aspect in floating body hydrodynamics is the steady oscillating motion of the structure at the free surface. The oscillations of a floating body were the first consistently examined by studying the superimposition of wave harmonics [1], and the transient motions of a freely floating structure that simultaneously solved the time-domain fluid-motion and rigid-body dynamics of the system [2]. Since then many researchers have theortically examined the problem using methods like Galerkin approximation and finite element formulation [3], classical matched eigenfunction expansion method [4][5], Boussinesq-type equations and eigenmode expansion [6] etc. These and other works collectively expounded the hydrodynamics of such systems using scattered, radiated and diffracted potentials, as well as wave forces, surface elevations and hydrodynamic coefficients. In spite of the large amount of literature on floating breakwaters in numerical, analytical and experimental approaches, the growing complexity of current and upcoming models has reignited the need for fast and stable numerical schemes that can instantaneously couple the fluid forces in the system and dynamics of an arbitrary floating body.

In the current work, the hydrodynamics of a general system of multiple floating structures, each having vertical oscillations is modeled using linear water wave theory. The simulation involves coupling the equations of motion for the floating body and the fluid potentials. The boundary value problem is solved using hybrid-element method combining eigenfunction expansion and boundary element methods (EFEM-BEM). The hydrodynamics are studied using frequency dependent coefficients. Results for multiple box type floating breakwaters are presented in this scenario. In addition to reducing wave load excitation on the floating structures, the study will enhance the understanding of the impacts of various physical parameters for minimizing incident wave energy, wave reflection, and transmission.

2. MATHEMATICAL FORMULATION

Wave interaction with multiple floating structures is modeled using the linear water wave theory and solved using a hybrid element method EFEM-BEM. The fluid-structure interaction is considered in a cartesian co-ordinate system wherein the fluid is modeled as inviscid, incompressible and irrotational, due to which a scalar velocity potential $\Phi(x, y, z, t)$ can be defined satisfying $\nabla^2\Phi = 0$. The configuration in Fig. 1 consists of N floating structures A_j having different widths r_j, where $j = 1, 2, \cdots N$, over a uniform seabed at water depth $z = -h$. The spacing between the structures j and $j+1$ is l_j, and the distances d_j are defined recursively as $d_1 = r_1,\ d_{j+1} = (d_j + l_j + r_{j+1}),\ j = 2, \cdots, (N-1)$. Thus, the fluid domain under consideration is divided into the water region A_0, with outer regions one the sea- and lee- side labelled as O_1 and O_2, respectively. It is presumed that the incident wave makes an angle θ, with the $x-$ axis and the fluid flow is simple harmonic in time with angular frequency ω. The existing velocity potential in the three regions is of the form $\Phi(x, z, t) = \Re\{\phi(x, z)e^{-\mathrm{i}(k_y y - \omega t)}\}$, and $k_y = k_0 \sin\theta$, where k_0 is the progressive wave number, and the spatial velocity potentials $\phi(x, z)$ are solutions to the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - k_y^2\right)\phi = 0 \tag{1}$$

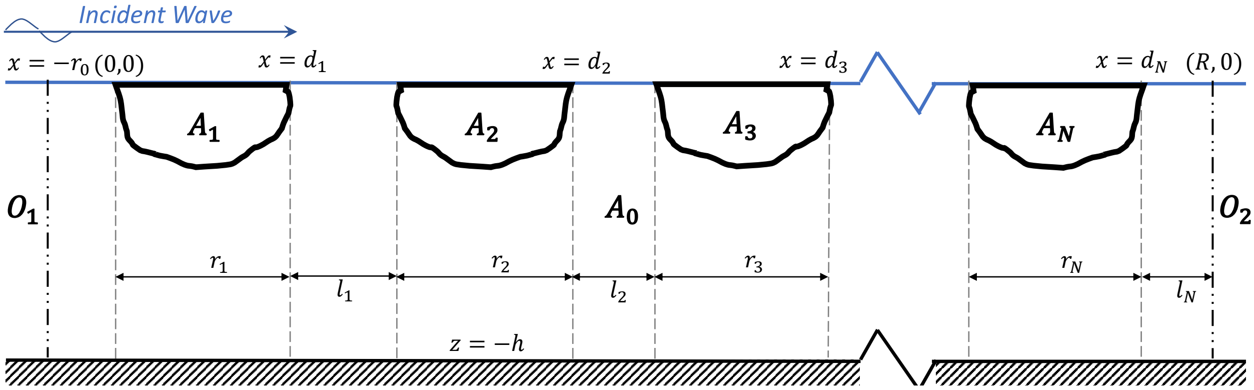


Figure 1: Schematic for the wave scattering configuration of multiple floating structures over a uniform seabed.

The mean free surface is governed by kinematic and dynamic boundary conditions which are combined to give the linearized free surface boundary condition as

$$\frac{\partial\phi}{\partial z} - K\phi = 0,\ \text{for } z = 0 \text{ in } A_0, O_1, O_2, \tag{2}$$

where $K = \omega^2/g$, and the bottom boundary condition on the rigid sea floor is

$$\frac{\partial\phi_j}{\partial z} = 0,\ \text{for } z = -h \text{ in } A_0, O_1, O_2. \tag{3}$$

On the fluid-structure interface S_j between structure A_j and water region A_0, $j = 1, 2, \cdots N$, assuming the uniform vertical displacement η_j, the oscillatory motion is governed by the following N pair of equations for $j = 1, 2, \cdots N$

$$\frac{\partial\phi}{\partial \boldsymbol{n}} = \frac{\partial\eta_j}{\partial t}(\boldsymbol{k}.\boldsymbol{n}) \tag{4}$$

$$m_j\frac{\partial^2\eta_j}{\partial t^2} = -\rho_j g\eta_j + i\omega\rho_j\int_{S_j}\frac{\partial\phi}{\partial t}(\boldsymbol{k}.\boldsymbol{n})\ ds, \tag{5}$$

where $\boldsymbol{k}$ is the unite vector in z-direction, $\boldsymbol{n}$ is the outward normal to the region A_0, ρ_j is the structure density, and m_j is the structure mass. The radiation conditions for outer regions O_1 and O_2 are

$$\phi_{O_1} = \left(I_0 e^{-\mathrm{i}q_0 x} + \mathscr{R}_0 e^{\mathrm{i}q_0 x}\right)\zeta_0(k_0, z) \quad \text{as} \quad x \to -\infty, \tag{6}$$

$$\phi_{O_2} = \mathscr{T}_0 e^{-\mathrm{i}q_0 x}\zeta_0(k_0, z) \qquad \text{as} \quad x \to \ \infty, \tag{7}$$

where I_0 is the known incident amplitude, $\mathscr{R}_0$ and $\mathscr{T}_0$ are the undetermined constants. $\zeta_0(k_0, z)$ is the eigenfunction in the outside region O_1 with $q_0 = \sqrt{k_0^2 - k_y^2}$.

3. SOLUTION METHODOLOGY

The numerical modeling of the wave action against N floating structures is undertaken using the hybrid element method (EFEM-BEM) applied on the three domains. The above boundary value problem is tackled using the eigenfunction expansion of the potential in the outer regions O_1, O_2 combined with a boundary element method in the inner fluid domain. The fundamental solution Green's function, derived using Green's theorem, is implemented to convert the BVP into a boundary integral equation. The domain of infinite water region is constricted by using fictitious auxiliary boundaries defined at $x = -r_0$ and $x = R = d_N + l_N$ where r_0 and l_N are defined for auxiliary boundaries on left and right of the $N-$structure array. The velocity potentials from eigenfunction expansion method are used to evaluate the boundary conditions on auxiliary boundaries and subsequently used in the boundary element formulation that begins with $\mathcal{G}_0 \equiv \mathcal{G}(\boldsymbol{x}, \boldsymbol{x}_0)$ as the fundamental solution to Laplace equation when it satisfies $\nabla^2 \mathcal{G}_0 + \delta_{\boldsymbol{x}_0} = 0$, where $\delta_{\boldsymbol{x}_0}$ is the Dirac delta distribution at the source point $\boldsymbol{x}_0$. Thus $\mathcal{G}_0$ represents a radially symmetric solution field emanating from a unit source concentrated at $\boldsymbol{x}_0$ and extending to infinity. Implementing the fundamental solution $\mathcal{G}_0$ and the potential ϕ into the *Green's* 2^{nd} *identity*, the boundary integral representation can be formalized as follows:
Let ϕ be a harmonic function in $\Gamma \in \mathbb{R}^2$, a simple open and connected domain with boundary $\partial\Gamma$. Then $\phi(\boldsymbol{x}_0)$ for $\boldsymbol{x}_0 \in \Omega$ can be expressed as

$$\phi(\boldsymbol{x}_0) = \int_{\partial\Gamma} \left(\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}_0) \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{n}} - \phi(\boldsymbol{x}) \frac{\partial \mathcal{G}(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial \boldsymbol{n}} \right) d\boldsymbol{x} \tag{8}$$

where $\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}_0)$ is the fundamental solution of the Laplace equation and

$$\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{2} \ln |\boldsymbol{x} - \boldsymbol{x}_0|_2 \quad \text{and} \quad \frac{\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial \boldsymbol{n}} = -\frac{\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_0)}{||\boldsymbol{x} - \boldsymbol{x}_0||_2^2} \tag{9}$$

In case of oblique waves, the the fundamental solution to the Helmholtz equation Eq. (1) is Green's function $\mathcal{G}$ satisfying

$$\frac{\partial^2 \mathcal{G}}{\partial x^2} + \frac{\partial^2 \mathcal{G}}{\partial z^2} - k_y^2 \mathcal{G} = \delta\left(x - x_0\right) \delta\left(z - z_0\right), \text{ given by} \tag{10}$$

$$\mathcal{G}\left(x,\, z\,;\, x_0,\, z_0\right) = -\frac{\mathrm{K}_0\left(k_y r\right)}{2\pi}, \quad \text{and} \quad \frac{\partial \mathcal{G}}{\partial \boldsymbol{n}} = \frac{k_y}{2\pi} \mathrm{K}_1(k_y r) \frac{\partial r}{\partial \boldsymbol{n}}, \tag{11}$$

where $r = \sqrt{\left(x - x_0\right)^2 + \left(z - z_0\right)^2}$ is the distance between (x_0, z_0) and (x, z) (i.e. source point to field point), K_0, K_1 are the second kind modified Bessel functions of 0^{th} order, 1^{st} order, respectively. In both cases of normal and oblique waves, for a closed domain Γ having boundary $\partial\Gamma$ the boundary integral representation from Eq. (8) is given by

$$\int_{\partial\Gamma} \left(\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}_0) \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{n}} - \phi(\boldsymbol{x}) \frac{\partial \mathcal{G}(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial \boldsymbol{n}} \right) d\boldsymbol{x} = \begin{cases} 0, & \boldsymbol{x}_0 \text{ outside } \Gamma, \\ \frac{1}{2}\phi(\boldsymbol{x}_0), & \boldsymbol{x}_0 \text{ on } \partial\Gamma, \\ \phi(\boldsymbol{x}_0), & \boldsymbol{x}_0 \text{ inside } \Gamma, \end{cases} \tag{12}$$

Eqs. (12) are solved for the entire boundary $\partial\Gamma$ of the three domains using constant element boundary element method (BEM). The physical boundaries of the domains $\partial\Gamma_j$ are discretized into finite number of elements on which the values of ϕ and $\phi_{\boldsymbol{n}}$ are taken as constant. Implementing the boundary conditions, the integral equations are discretized, following which the kernels are integrated using Gaussian integration to evaluate the integral coefficients. The resulting system of equations is solved as a matrix equation and the unknown potentials are evaluated.

3.1 Velocity potentials in the outer region using eigenfunction expansion

In particular, due to the absence of a structure in the outer regions O_1 and O_2 and approximation of a uniform bottom bed bottom bed, the velocity potentials are deduced by the eigenfunction expansion method, such that for $k_n,\ n = 1, 2, \cdots$ modes, the spatial velocity potential satisfying Eq. (1) along with Eqs. (2)-(3) and Eqs. (6)-(7) in the outer region are given by a combination of eigenfunctions and expressed as

$$\phi_{O_1} = I_0 e^{-\mathrm{i}q_0(x+r_0)}\zeta_0(k_0, z) + \sum_{n=0}^{\infty} \mathscr{R}_n\ e^{\mathrm{i}q_n(x+r_0)}\zeta_n(k_n, z) \quad \text{at} \quad \Gamma_L, \tag{13}$$

$$\phi_{O_2} = \sum_{n=0}^{\infty} \mathscr{T}_n\ e^{-iq_n(x-R)}\zeta_n(k_n, z) \quad \text{at } \Gamma_R, \tag{14}$$

where $\zeta_n(k_n, z)$ are given by $\zeta_n(k_n, z) = \left(\frac{\mathrm{i}g}{\omega}\right)\frac{\cosh k_n(z+h)}{\cosh k_n h}$, with k_n satisfying the dispersion relation $\omega^2 = gk_n \tanh(k_n h)$. The undetermined constants $\mathscr{R}_n$ and $\mathscr{T}_n$ are evaluated using orthogonality of eigenfunctions ζ_n in the outside regions as

$$\mathscr{R}_n + \delta_{n0} = \frac{1}{\mathscr{E}_n}\int_{-h}^{0} \phi_{O_1}(z)\Big|_{x=-r_0} \zeta_n(k_n, z)\, dz, \tag{15}$$

$$\mathscr{T}_n = \frac{1}{\mathscr{E}_n}\int_{-h}^{0} \phi_{O_2}(z)\Big|_{x=R} \zeta_n(k_n, z)\, dz, \tag{16}$$

where $\mathscr{E}_n = \int_{-h}^{0} {\zeta_n}^2(k_n, z)\, dz = -\left(\frac{g^2}{\omega^2}\right)\frac{k_n h + \sinh(k_n h)\cosh(k_n h)}{2k_n \cosh^2(k_n h)}$ and δ_{n0} is the Kronecker delta function. The derivative of potential is given as

$$\frac{\partial \phi_{O_1}}{\partial \boldsymbol{n}}\Big|_{x=-r_0} = -\frac{\partial \phi_{O_1}}{\partial x}\Big|_{x=-r_0} = \mathrm{i}q_0 I_0 \zeta_0(k_0, z) - \sum_{n=0}^{\infty} \mathrm{i}q_n \mathscr{R}_n \zeta_n(k_n, z). \tag{17}$$

Substituting for $\mathscr{R}_n$ from Eq.(15) in Eq.(17) and truncating the infinite series after N-terms, it is derived that

$$\frac{\partial \phi_{O_1}}{\partial \boldsymbol{n}}\Big|_{x=-r_0} = \Omega\left[\phi_{O_1}(z)\Big|_{x=-r_0}\right] + 2\mathrm{i}q_0 I_0 \zeta_0, \text{ where } \Omega \text{ is given by} \tag{18}$$

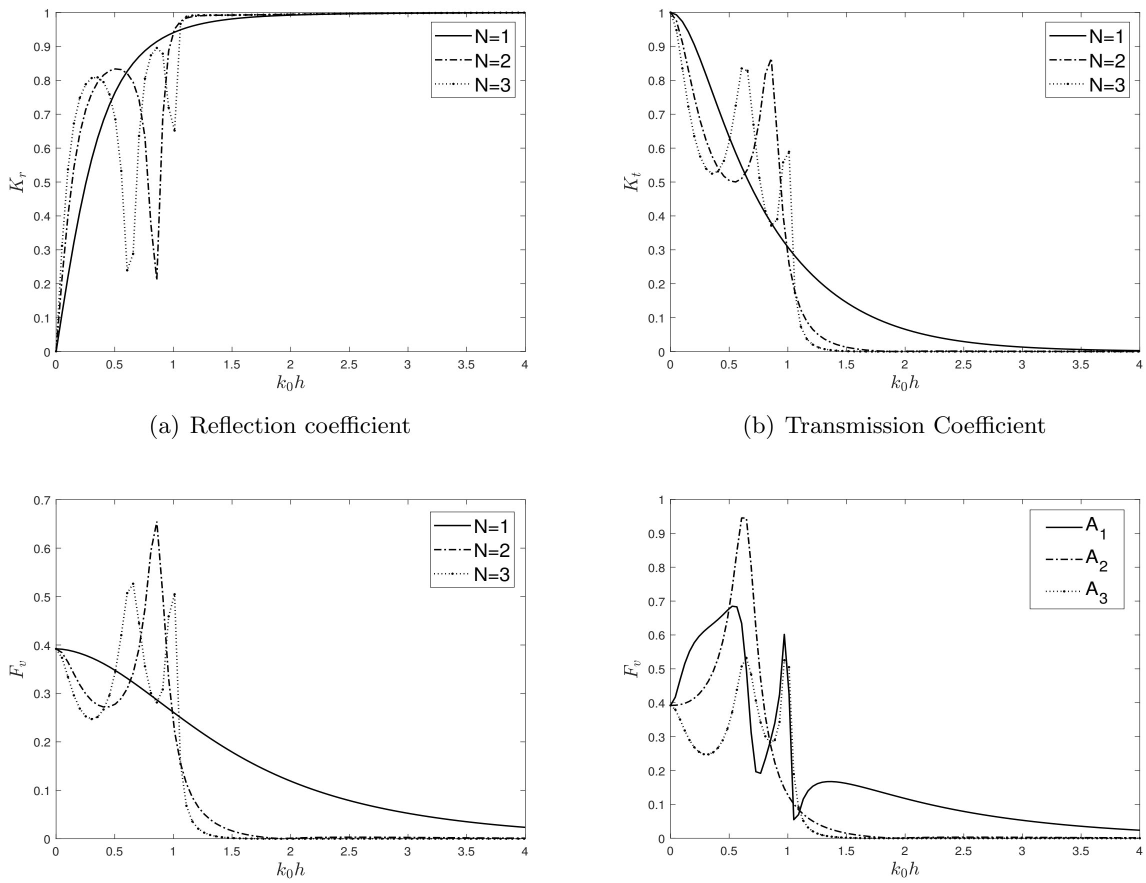
$$\Omega\left[\phi_{O_1}(z)\Big|_{x=-r_0}\right] = -\sum_{n=0}^{N} \frac{\mathrm{i}q_n}{\mathscr{E}_n}\left\{\int_{-h}^{0} \phi_{O_1}(s)\,\zeta_n(k_n, s)\, ds\right\} I_n(k_n, z). \tag{19}$$

The integral $\int_{-h}^{0} \phi_{O_1}(s)\,\zeta_n(k_n, s)\, ds$ is evaluated using the same boundary element discretization as in the evaluation of Green's function and its normal derivative on the auxiliary boundary, which in turn gives the matrix factorization of Ω. Thus, the integral is evaluated exactly, which in turn gives the matrix factorization of Ω. Finally, Eqs. (12) and (19) are solved to derive a final matrix equation to determine the unknown constants of the outer solution.

4. RESULTS

In the following results, upto three structures of rectangular shape are considered. The optimal parameters $r_j/h = 0.5, l_j/h = 0.5, \forall j$ and $\theta = 10°$, kept fixed, unless otherwise stated. The observations here are mainly discussed for wave scattering by the system and wave forces on the multiple structures in varied configurations. The reflection, transmission and dissipation coefficients (K_r, K_t and K_d) and vertical wave force on structure A_j are calculated as

$$K_r = \left|\frac{\mathscr{R}_0}{I_0}\right|, \quad K_t = \left|\frac{\mathscr{T}_0}{I_0}\right|, \quad K_d = 1 - K_r^2 - K_t^2, \quad F_v = \frac{i\omega}{gh^2}\int_{S_j} \phi(\boldsymbol{k}.\boldsymbol{n})\ ds. \tag{20}$$



(a) Reflection coefficient
(b) Transmission Coefficient
(c) Vertical force on lee-side structure
(d) Vertical force on each structure

Figure 2: Frequency dependence of hydrodynamics coefficients for multiple floating structures.

The above figures illustrate the frequency dependence of the hydrodynamic coefficients by plotting K_r, K_t and F_v against non-dimensional wavenumber k_0h. It is noticeable that with increase in the number of structures, the oscillatory patterns in the reflection and transmission coefficients increase (Figs.2(a),2(b)), indicating interference of incoming, reflected and radiated waves. This is also indicated by the vertical wave force on the lee side structure (Fig.2(c)) and wave forces on different structures (Fig.2(d)). The peaks and troughs in F_v correspond to interference of the waves that result in increasing oscillations in the floating bodies. These results agree with those already found in the literature, signifying the accuracy of the proposed method.

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