Reduced Order Modelling for Dispersive and Nonlinear Water Wave Modelling

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1 INTRODUCTION

This abstract describes our recent work on employing reduced-order modelling (ROM) to solve fully nonlinear potential flow equations (FNPF) to achieve faster turn-around time than a full order model (FOM) based on the spectral element method (SEM) [1]. We propose a POD-Galerkin based model-order reduction approach to reduce the cost of the solve step in the Laplace problem. If repeated simulations are needed for applications, e.g. in optimization loops with varying parameters, it may become prohibitively expensive to run many FOM simulations in practical times. Reduced-order modelling techniques were introduced to eliminate the time-consuming behaviour of high-dimensional numerical methods and reduce the load on computational resources without compromising overall accuracy. The proper orthogonal decomposition (POD) method is one of the most effective snapshot-based reduced-order modelling techniques and is considered in this work. The basic idea of using POD is to generate a low-dimensional model with few degrees of freedom using the most dominant features of the system, thereby significantly reducing the computational time and cost.

2 MATHEMATICAL FORMULATION

We consider the Eulerian FNPF model, cf. complete derivation in [2]. Mass continuity is expressed in terms of the Laplace equation for the scalar velocty potential ϕ within the fluid volume (Ω) subject to a free surface condition (Γ^{FS}) and impermeable walls and sea floor (Γ^b)

$$\phi = \tilde{\phi}, \quad z = \eta \quad \text{on} \quad \Gamma^{FS}, \tag{1a}$$

$$\nabla^2 \phi = 0, \quad -h(x,y) < z < \eta \quad \text{in} \quad \Omega, \tag{1b}$$

$$\mathbf{n} \cdot \nabla \phi = 0, \quad z = -h(x,y) \quad \text{on} \quad \Gamma^b, \tag{1c}$$

and subject to the nonlinear free surface boundary conditions in the time domain (T)

$$\frac{\partial \eta}{\partial t} = -\boldsymbol{\nabla}\eta \cdot \boldsymbol{\nabla}\tilde{\phi} + \tilde{w}(1 + \boldsymbol{\nabla}\eta \cdot \boldsymbol{\nabla}\eta) \qquad \text{in} \quad \Gamma^{\mathrm{FS}} \times T, \tag{2a}$$

$$\frac{\partial \tilde{\phi}}{\partial t} = -g\eta - \frac{1}{2}\Big(\boldsymbol{\nabla}\tilde{\phi} \cdot \boldsymbol{\nabla}\tilde{\phi} - \tilde{w}^2(1 + \boldsymbol{\nabla}\eta \cdot \boldsymbol{\nabla}\eta)\Big) \quad \text{in} \quad \Gamma^{\mathrm{FS}} \times T, \tag{2b}$$

where the '$\sim$' symbol is used for free surface (FS) variables. The equations are widely used to describe both free surface flow as well as wave-structure interaction and have been subject to many investigations for decades when it comes to the design of flexible and efficient numerical schemes. While the numerical discretization of FNPF equations is well-established, in recent years, renewed attention has been given to make progress with CFD solvers, where FNPF models as medium fidelity models are demonstrated to be several orders of magnitude faster than CFD solvers for non-breaking wave and wave-structure simulations, and couplings of these types of solvers can be exploited for enabling high-fidelity CFD-FNPF simulations.

3 NUMERICAL DISCRETIZATION

The numerical discretization of the model equations (1) is based on the σ-transformed SEM scheme detailed in [1] (with the $\mathcal{K}$-matrix defined for the σ-formulation). The numerical scheme for the Laplace equation is given here in brevity. Find $\phi(t,x,z)$ satisfying $\mathcal{L}\phi = \mathbf{b}$, where

$$\mathcal{L}_{ij} = -\sum_{k=1}^{N^k} \int_{\Omega^{c,k}} (\mathcal{K}(\eta(t))\nabla^c N_j) \cdot \nabla^c N_i \mathrm{d}\boldsymbol{x}, \quad b_i = \sum_{s=1}^{N^s} \oint_{\Gamma^{c,s}} N_i \mathbf{n} \cdot \left(\mathcal{K}(\eta(t))\nabla^c \left(\sum_{j=1}^{N^p} \phi_j N_j \right) \right) \mathrm{d}\boldsymbol{x}, \quad \text{(3a)}$$

for $i = 1, ..., N^p$. Here N^k is the number of elements, N^s is boundary segments, and $N^p = N^i + N^b$ is total grid points in the σ-transformed domain Ω^c. We use the decomposition

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{bb} & \mathcal{L}_{bi} \\ \mathcal{L}_{ib} & \mathcal{L}_{ii} \end{pmatrix}, \; \phi = \begin{pmatrix} \phi_b \\ \phi_i \end{pmatrix}, \; \mathbf{b} = \begin{pmatrix} b_b \\ b_i \end{pmatrix}, \quad \mathcal{L} \in \mathbb{R}^{N^p \times N^p}, \quad \mathbf{b} \in \mathbb{R}^{N^p}, \quad \text{(3b)}$$

where the subscripts b and i correspond to the free surface boundary and the inner variables, respectively. The system is defined in terms of the unknown inner values in the modified form

$$\mathcal{L}^*\phi_i = \mathbf{b}^*, \quad \mathcal{L}^* \in \mathbb{R}^{N^i \times N^i}, \quad \mathbf{b}^* \in \mathbb{R}^{N^i}, \quad \text{(3c)}$$

where $\mathcal{L}^* = \mathcal{L}_{ii} - \mathcal{L}_{ib}\mathcal{L}_{bb}^{-1}\mathcal{L}_{bi}$ and $\mathbf{b}^* = b_i - \mathcal{L}_{ib}\mathcal{L}_{bb}^{-1}b_b$. The Laplace problem has $b_i = 0$.

4 PROJECTION-BASED MODEL ORDER REDUCTION

Now, we present the essentials of the POD setting that is used to generate a reduced basis. For further details of the method, see [3]. First, POD starts with an ensemble of snapshots, $\{\phi(x,z,t_i)\}_{i=1}^{N^S}$. This is a set of FOM solutions obtained by solving the system of equations (3), at different time instances, $t_j = 1, 2, ..., N^S$, to the Laplace problem. These are collected in a snapshot matrix

$$\mathcal{S} = [\phi_{j,1}, \phi_{j,2}, ..., \phi_{j,N^S}], \quad \mathcal{S} \in \mathbb{R}^{N^p \times N^S}. \quad \text{(4)}$$

We then seek to generate a POD basis for constructing a reduced-order system from the snapshots matrix, $\mathcal{S}$, by using the singular value decomposition (SVD), $\mathcal{S} = \mathcal{U}\Sigma\mathcal{V}^T$. The aim is to express the solution in terms of a reduced basis using the eigenvectors of the SVD such that $\phi_i^l(x,z,t) = \mathcal{V}_l\mathbf{a}(t)$. For this ansatz, we use the N^l most important modes in the reduced basis, and construct an intrusive POD-Galerkin projection of the Laplace system. This gives rise to a reduced basis system of equations in the form

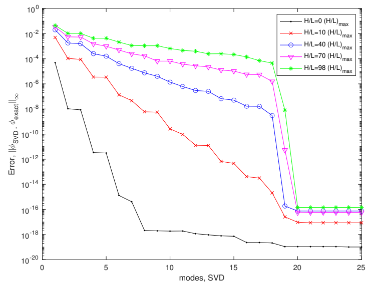
$$\mathcal{L}^\dagger\mathbf{a} = \mathbf{b}^\dagger, \quad \mathcal{L}^\dagger \in \mathbb{R}^{N^l \times N^l}, \quad \mathbf{b}^\dagger \in \mathbb{R}^{N^l}, \quad \text{(5)}$$

where the system is defined from $\mathcal{L}^\dagger = \mathcal{V}_l^T\mathcal{L}^*\mathcal{V}_l$ and $\mathbf{b}^\dagger = \mathcal{V}_l^T\mathbf{b}^*$. Hence, it is possible to solve the reduced system of equations which from snapshot data learns basis functions that capture the variations in the solution with respect to time when the free surface boundary condition is changing.

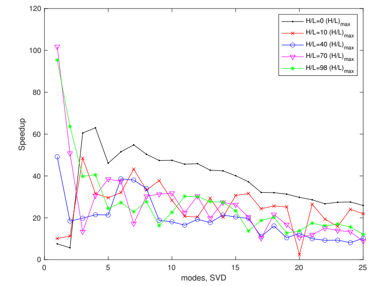
Remark, the reduced system matrix (5) is dense, and hence it becomes intrinsically difficult to scale the system degree of freedom (N^p) to large-scale as is possible in a p-multigrid accelerated FNPF-SEM solver [4]. Hence, the advantage of acceleration of the FNPF-SEM solver lies in the ability to reduce the cost of simulation for certain problem sizes and parametrizations. From the solution to the linear system (5), the FOM solution is estimated $\phi_i \approx \phi_i^l$ after which the vertical free surface velocity $\tilde{w}$ is computed in the usual way from ϕ, recalling $(u,w) = (\partial_x, \partial_z)\phi$, to obtain closure in the free surface time-stepping problem (2).

5 NUMERICAL STUDIES

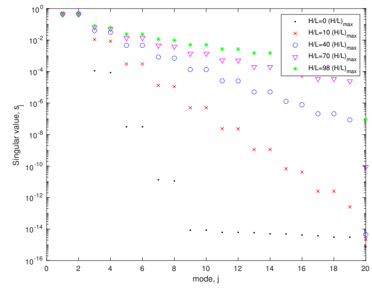
This section presents partial results of our numerical experiments for accelerating the FNPF-SEM solver through using POD-Galerkin for the Laplace solve step. Our first numerical experiment assesses the potential of using ROM for acceleration of the Laplace problem by considering nonlinear stream function waves. This is a relevant benchmark for nonlinear wave propagation subject to variations in the dispersion (kh) and nonlinear wave (H/L) parameters. Here $k = 2\pi/L$ is wave number, L is wave length, and H is wave height. In Figures 1a-1c, we present results of a convergence test, where a fixed spatial resolution is used with N^p =144 nodes in the mesh. The elements are quadrilaterals with expansions of polynomial order $P = 6$, meshed in terms of a single vertical layer and four elements in the horizontal direction. We vary the nonlinearity of the waves such that $H/L = 0.1, 10, 40, 70, 98$ % of the theoretical $(H/L)_{max}$. From sampling the variation of the free surface evolution across a wave period, a basis is constructed using the POD procedure in each case. More than 99% of the variation in the waves are captured with few modes ($N^l < 15$) in each case due to a strong exponential decay in the singular modes. This translates into a possible acceleration of up to about x100 times of the Laplace solve step. Similar results are given in Figures 1d-1f for a higher resolution case with a mesh



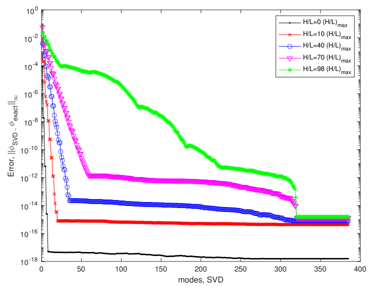
(a) Convergence tests, ϕ, $kh = 2\pi.(N_x^k, N_\sigma^k) = (4, 1)$.



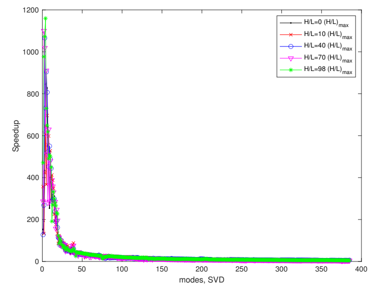
(b) Speedup estimates, $kh = 2\pi.\ (N_x^k, N_\sigma^k) = (4, 1)$.



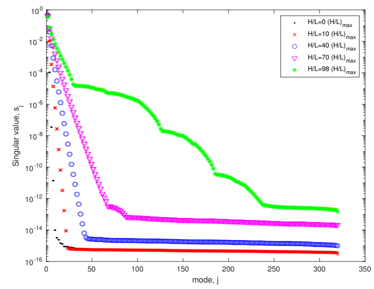
(c) Singular values, $kh = 2\pi.\ (N_x^k, N_\sigma^k) = (4, 1)$.



(d) Convergence tests, ϕ, $kh = 2\pi.\ (N_x^k, N_\sigma^k) = (64, 16)$.



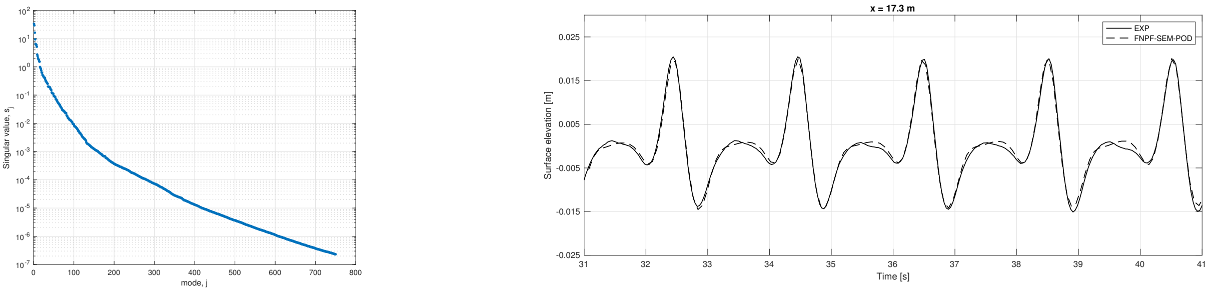
(e) Speedup estimates, $kh = 2\pi.\ (N_x^k, N_\sigma^k) = (64, 16)$.



(f) Singular values, $kh = 2\pi.\ (N_x^k, N_\sigma^k) = (64, 16)$.

Figure 1

consisting of N^p =36864 nodes in a layered configuration with $(N_x^k, N_\sigma^k) = (64, 16)$ elements. The potential speedup is for this setup up to 2-3 orders of magnitude in the low accuracy tolerance level regime, and with increasing potential the more resolution in the vertical as it is most important to capture details near the free surface. The former results presented build up to testing a POD-Galerkin ROM for a more realistic case. We consider the experiments due to Beji & Battjes (1994) of harmonic generation resulting from dispersive and nonlinear waves passing a submerged bar inside a numerical wave tank in 2D as the second experiment. The setup is based on Case A of the experiments. The input wave is generated in the numerical wave tank using a stream function solution at undisturbed depth 0.4m with a wave height of 2cm and a wave period of 2.02s. In this setup, the FOM model uses a configuration



(a) (b)

Figure 2: Harmonic generation over a submerged bar in 2D. (a) Singular values of the snapshot matrix, and (b) numerical results obtained from the FNPF-SEM-ROM model at gauge 6.

$(N_x^k, N_\sigma^k, P) = (103, 1, 6)$, and snapshots are generated by running a simulation until t_{final}=20s. The final FNPF-SEM-ROM results are obtained by then running the entire simulation until t_{final}=41s with wave generation and absorption. When the basis has been generated using a POD offline, we run the same simulation online with only N^l =500 modes in the reduced system (5). Results are presented in Figure 2b for Gauge 6 at 17.3m in the experiments using the setup from [1]. Remark, the speedup potential is currently reduced, since the computational complexity of constructing the Laplace operator is similar to FOM simulations as it is updated through a global assembly before projection at every RK stage of en ERK4 scheme for the time-integration due to the time-dependent coefficients of the σ-transformed Laplace equation (3a).

CONCLUSIONS

The results obtained from numerical experiments demonstrate that the potential speedup is influenced by both dispersion and nonlinearity parameters as expected. For the most nonlinear waves, the rank of the SVD increases, requiring more POD modes and lowering the potential speedup. However, our tests indicate a significant reduction of system size is possible in the range of linear to mildly nonlinear waves and for problems, where there is a need for significant spatial resolution in the vertical. The latter suggests a significant potential for acceleration of wave-structure problems, where the POD-Galerkin approach can be used to account also for the parametrisation of geometric features of structures.

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