Time dependent motion due to sinusoidal ocean bottom oscillation

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1 INTRODUCTION

Coastal areas, including land-mass and water, have always been a significant part of human civilisation. More than 600 million people live in a coastal region within 10 m of elevation, which accounts for 10 per cent of the world population. On top of that, approximately 2.4 billion people, around 40 per cent of the total world population, live within 100 Kms of the coastline. Any threat to the coastal region thus possesses an enormous amount of potential damage to the world population, a country's economy, natural food resources, to name a few. Among all the natural calamities, Tsunamis possess a grave threat which is evident from the recent accounts of destruction caused by 2004 Indian Ocean, 2011 Tohoku Oki, 2018 Sulawesi and Palu Tsunami, which were caused following submarine earthquakes. Predicting and simulating Tsunami waves is of obvious importance in providing a reliable warning system. In particular, the inclusion of compressibility allows for the simulation of acoustic gravity waves, which have been proposed as a method for early warning [1, 2, 3]. It has also been shown that a Tsunami model based on a compressible ocean is more accurate than a model with an incompressible ocean [2]. Many other authors have shown this [4, 5, 2, 6]. It is also essential to simulate different kinds of motions of the ocean bottom. An appropriate model taking the water compressibility into account and that can cater to such changes in the initial time-domain displacement is developed in this work.

2. MATHEMATICAL FORMULATION

We consider free-surface gravity wave propagation in a compressible ocean of finite depth h. The physical problem is formulated in a two-dimensional Cartesian coordinate system having z- axis pointing upwards and x- axis horizontal. The ocean bed is characterised as rigid. A wave propagation due to the ocean floor disturbance is realised both towards the positive and negative x- direction under the assumption of linearised water wave theory. The flow is considered irrotational. We are interested in calculating the time-dependent motion of the fluid due to a movement of the seafloor, simulating the generation of a Tsunami in two dimensions. We consider time-dependent growth $l(t)$ of a fixed displacement function $\mathcal{X}(x)$ of the ocean bottom between $-b$ and b. We note that it would be

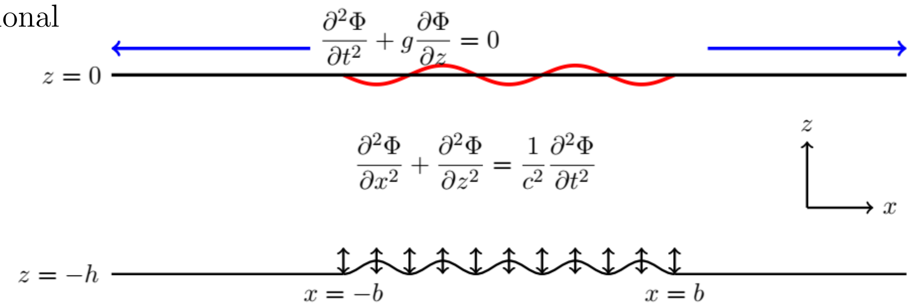

Figure 1: Schematic diagram of the single frequency problem with sinusoidal oscillating bottom profile.

straightforward to generalise to more complex motions which were not separable using linearity. Under these assumptions, the small amplitude displacement of the seafloor $\tilde{h}$ is given by

$$\tilde{h}(x,t) = l(t)\mathcal{X}(x)\mathcal{H}(b^2 - x^2),$$

where Φ is the displacement potential, g is the acceleration due to gravity, $c = \dfrac{K_0}{\rho_0}$ is the speed of sound in water, K_0 being the bulk modulus and ρ_0 is the undisturbed density of the whole water region, and $\mathcal{H}(\cdot)$ represents Heaviside unit step function. The boundary value problem we wish to solve is given by

$$\nabla^2\Phi(x,z,t) = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \qquad \text{in} \quad -h < z < 0, \tag{1a}$$

$$\Phi_{tt} + g\Phi_z = 0 \qquad \text{at} \quad z = 0, \tag{1b}$$

$$\Phi_z = l(t)\mathcal{X}(x)\mathcal{H}(b^2 - x^2) \qquad \text{at} \quad z = -h. \tag{1c}$$

We calculate the single frequency solution considering a sinusoidal bottom profile. We consider a block of the ocean bottom between $-b$ and b with a sinusoidal surface which is oscillating vertically with a unit amplitude and angular frequency ω (see Fig. 1) in the following form:

$$\zeta^m(x,t) = \exp\left(\mathrm{i}\frac{m\pi x}{b}\right)\mathcal{H}(b^2 - x^2)e^{\mathrm{i}\omega t} = \zeta_c^m(x,t) + \mathrm{i}\zeta_s^m(x,t), \quad m \in \mathbb{Z} \text{ (set of integers)}$$

where

$$\zeta_c^m(x,t) = \cos\left(\frac{m\pi x}{b}\right)\mathcal{H}(b^2 - x^2)e^{\mathrm{i}\omega t}, \quad \text{and} \quad \zeta_s^m(x,t) = \sin\left(\frac{m\pi x}{b}\right)\mathcal{H}(b^2 - x^2)e^{\mathrm{i}\omega t}, \tag{2}$$

In this case the displacement potential can be written as $\Phi(x,z,t) = \phi^m(x,z)\exp(\mathrm{i}\omega t)$ and ϕ^m satisfies the following equations

$$\nabla^2\phi^m(x,z,) = -\frac{\omega^2}{c^2}\phi^m \qquad \text{in} \quad -h < z < 0,$$

$$-\omega^2\phi^m + g\phi_z = 0 \qquad \text{at} \quad z = 0,$$

$$\phi_z^m = \exp\left(\mathrm{i}\frac{m\pi x}{b}\right)\mathcal{H}(b^2 - x^2) \qquad \text{at} \quad z = -h.$$

The problem will be solved by applying Fourier transformation of the form

$$\mathcal{F}(k,z) = \int_{-\infty}^{\infty}\phi^m(x,z)\exp(-\mathrm{i}kx)dx, \tag{4}$$

whose inverse Fourier transformation is given by

$$\phi^m(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(k,z)\exp(\mathrm{i}kx)dk, \tag{5}$$

The potential $\phi(x,z,\omega)$ is obtained as

$$\begin{aligned}
\phi^m(x,z,\omega)|_{|x|<b} = b(-1)^m &\left\{ \frac{4\mu_0\cosh\mu_0(z+h)e^{-\mathrm{i}k_0 b}(m\pi\mathrm{i}\sin(k_0x) + k_0b\cos(k_0x))}{k_0(m^2\pi^2 - k_0^2b^2)(2\mu_0h + \sinh 2\mu_0h)} \right. \\
&+ \sum_{n=1}^{N}\frac{4\mu_n\cos\mu_n(z+h)e^{-\mathrm{i}k_nb}(m\pi\mathrm{i}\sin(k_nx) + k_nb\cos(k_nx))}{k_n(m^2\pi^2 - k_n^2b^2)(2\mu_nh + \sin 2\mu_nh)} \\
&\left. + \sum_{n=N+1}^{\infty}\frac{4\mu_ne^{-\lambda_nb}\cos(\mu_n(z+h))(im\pi\sinh(\lambda_nx) + \lambda_nb\cosh(\lambda_nx))}{\lambda_n(m^2\pi^2 + \lambda_n^2b^2)(2\mu_nh + \sin(2\mu_nh))} \right\} \\
&- (-1)^m\frac{\xi_m\cosh\xi_mz + (\omega^2/g)\sinh\xi_mz}{\xi_m\mathcal{D}(\xi_m,h)}e^{im\pi(x+b)/b}, \text{ where } \xi_m = \sqrt{\frac{m^2\pi^2}{b^2} - \frac{\omega^2}{c^2}}.
\end{aligned} \tag{6}$$

The solution for cosine type bottom ($\cos(m\pi x/b)$) can be retrieved as

$$\phi_c^m(x,z,\omega) = \frac{\phi^m(x,z,\omega) + \phi^{-m}(x,z,\omega)}{2}.$$

The region-wise explicit forms for the potential function turn out to be

$$\phi_c^m(x,z,\,\omega)|_{|x|>b} = b^2(-1)^m \left\{ \frac{-4\mathrm{i}\mu_0 \sin(k_0 b)\cosh(\mu_0(z+h))e^{\mp \mathrm{i}k_0 x}}{(m^2\pi^2 - k_0^2 b^2)(2\mu_0 h + \sinh(2\mu_0 h))} \right.$$
$$\left. + \sum_{n=1}^{N} \frac{-4\mathrm{i}\mu_n \sin(k_n b)\cos(\mu_n(z+h))e^{\mp \mathrm{i}k_n x}}{(m^2\pi^2 - k_n^2 b^2)(2\mu_n h + \sin(2\mu_n h))} - \sum_{n=N+1}^{\infty} \frac{4\mu_n \sinh(\lambda_n b)\cos(\mu_n(z+h))e^{\mp\lambda_n x}}{(m^2\pi^2 + \lambda_n^2 b^2)(2\mu_n h + \sin(2\mu_n h))} \right\} \tag{7}$$

The upper sign is for $x > b$ and the lower sign is for $x < -b$. When $|x| < b$, the following form of $\phi_c^m(x,z,\omega)$ is obtained:

$$\phi_c^m(x,z,\omega)|_{|x|<b} = b^2(-1)^m \left\{ \frac{4\mu_0 \cosh(\mu_0(z+h))e^{-\mathrm{i}k_0 b}\cos(k_0 x)}{(m^2\pi^2 - k_0^2 b^2)(2\mu_0 h + \sinh(2\mu_0 h))} \right.$$
$$\left. + \sum_{n=1}^{N} \frac{4\mu_n \cos(\mu_n(z+h))e^{-\mathrm{i}k_n b}\cos(k_n x)}{(m^2\pi^2 - k_n^2 b^2)(2\mu_n h + \sin(2\mu_n h))} + \sum_{n=N+1}^{\infty} \frac{4\mu_n e^{-\lambda_n b}\cos(\mu_n(z+h))\cosh(\lambda_n x)}{(m^2\pi^2 + \lambda_n^2 b^2)(2\mu_n h + \sin(2\mu_n h))} \right\}$$
$$- \frac{\xi_m \cosh \xi_m z + \frac{\omega^2}{g}\sinh \xi_m z}{\xi_m \left(\xi_m \sinh(\xi_m h) - \frac{\omega^2}{g}\cosh(\xi_m h) \right)} \cos\left(\frac{m\pi x}{b}\right), \quad \text{where} \quad \xi_m = \sqrt{\frac{m^2\pi^2}{b^2} - \frac{\omega^2}{c^2}}. \tag{8}$$

Similarly, the same for the sine type bottom ($\sin(m\pi x/b)$) can be obtained as

$$\phi_s^m(x,z,\omega) = \frac{\phi^m(x,z,\omega) - \phi^{-m}(x,z,\omega)}{2\mathrm{i}},$$

and the potential functions in the regions $|x| > b$ and $|x| < b$ can easily be computed. We now return to our primary problem of a finite time growth of the ocean bottom between $-b$ and b. We write $\mathcal{X}(x)$ as a Fourier series

$$\mathcal{X}(x) = \left(\sum_{m=0}^{\infty} \zeta_c^m \cos\left(\frac{m\pi x}{b}\right) + \zeta_s^m \sin\left(\frac{m\pi x}{b}\right) \right) \tag{9}$$

Applying a Fourier transformation in time (which is equivalent to a Laplace transform since the fluid is at rest)

$$\hat{\Phi}(x,z,\omega) = \int_0^{\infty} \Phi(x,z,t)e^{-\mathrm{i}\omega t}dt, \tag{10}$$

we can find the solution as

$$\hat{\Phi}(x,z,\omega) = \mathcal{W}(\omega)\left(\sum_{m=0}^{\infty} \zeta_c^m \phi_c^m(x,z,\omega) + \zeta_s^m \phi_s^m(x,z,\omega) \right) \text{ where } \mathcal{W}(\omega) = \int_0^{\infty} l(t)e^{-\mathrm{i}\omega t}dt.$$

Now taking the inverse Fourier transformation, we obtain the potential function as

$$\Phi(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\Phi}(x,z,\omega)e^{\mathrm{i}\omega t}d\omega = \mathrm{Re}\left\{ \frac{1}{\pi}\int_0^{\infty} \hat{\Phi}(x,z,\omega)e^{\mathrm{i}\omega t}d\omega \right\}. \tag{11}$$

We present here a formula for the surface elevation which is

$$\eta(x,t) = \frac{1}{\pi}\sum_{m=0}^{\infty}\zeta_m^c \mathrm{Re}\left\{\int_0^\infty \mathcal{W}(\omega,\tau) f_c^m(x,\omega) e^{\mathrm{i}\omega t} d\omega\right\} + \frac{1}{\pi}\sum_{m=1}^{\infty}\zeta_s^m \mathrm{Re}\left\{\int_0^\infty \mathcal{W}(\omega,\tau) f_s^m(x,\omega) e^{\mathrm{i}\omega t} d\omega\right\}.$$

The first integral in the RHS represents the surface elevation when $\cos(m\pi x/b)$ type bottom is considered, and the second term represents the same for $\sin(m\pi x/b)$ type bottom. The quantities $f_c^m(x,\omega)$ and $f_s^m(x,\omega)$ are defined by

$$f_c^m(x,\omega) = \left.\frac{\partial \phi_c^m(x,z,\omega)}{\partial z}\right|_{z=0} \quad \text{and} \quad f_s^m(x,\omega) = \left.\frac{\partial \phi_s^m(x,z,\omega)}{\partial z}\right|_{z=0}.$$

3 RESULTS

The surface profiles for a flat profile and for a Gaussian profile of the form $\mathcal{X} = \exp(-6(x/20000)^2)$ are shown for $\tau = 10$s, $l_{\max} = 1$, $b = 20$ Km and $h = 5$ Km in which

$$l(t) = \frac{l_{\max} t}{\tau}\mathcal{H}(t(\tau - t)) + l_{\max}\mathcal{H}(t-\tau) \quad \text{with} \quad \mathcal{W}(\omega,\tau) = l_{\max}\frac{\left(e^{-\mathrm{i}\omega\tau} - 1\right)}{\tau\omega^2}.$$

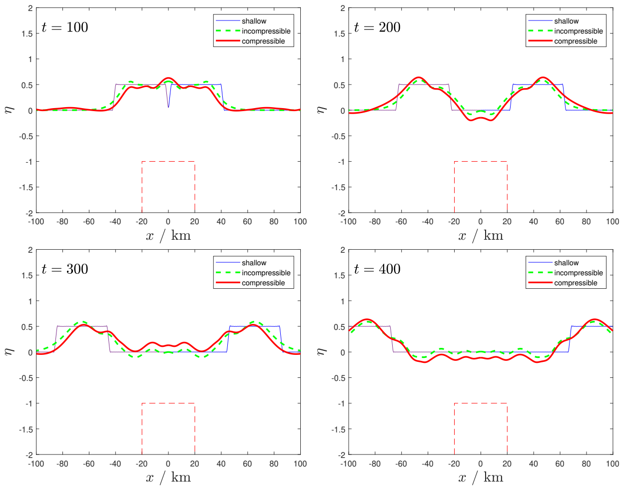

Figure 2: The bottom displacement is shown as a red dashed line for illustration only. The incompressible shallow water solution is also given along with the incompressible solution.

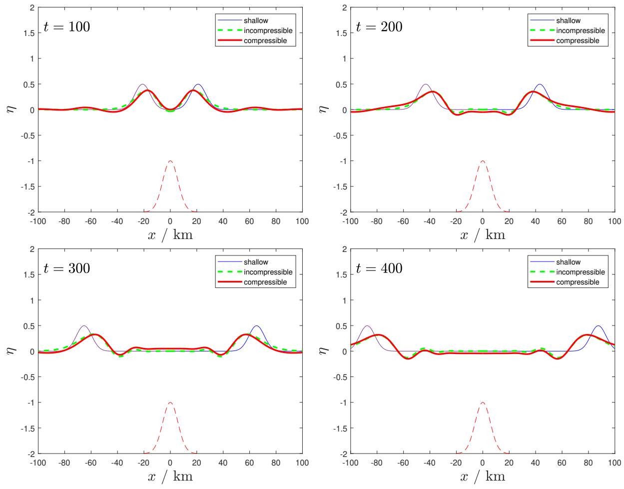

Figure 3: As in Fig. 2 except the bottom displacement is given by a Gaussian function.

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