Flexural gravity wave amplitudes in the vicinity of blocking point under shallow water approximation

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1 INTRODUCTION

Recently, there has been significant progress in studying the blocking of flexural gravity waves. Das et al. [1; 2] investigated the criteria for occurrences of flexural gravity wave blocking in homogeneous and two-layer fluids. Boral et al. [3] studied the wave amplitudes of flexural waves in the vicinity of the saddle as well as blocking points. Similarly, Barman et al. [4; 5] studied the wave scattering of flexural gravity wave due to ice-crack and bottom undulation in the context of wave blocking, and demonstrated the occurrence of removable discontinuities at the blocking points. In the present study, the characteristics of flexural gravity waves are studied near the saddle point as well as the blocking points for weakly varying compressive force in shallow water. The study reveals that the amplitude of flexural gravity waves in the vicinity of the saddle and blocking points satisfy the hyper-Airy and Airy differential equation, respectively, under the shallow water approximation. Using WKB theory, the asymptotic solution of the hyper-Airy differential equation is obtained for matching the far-field solution with the near field solution. Moreover, the simulation of time-dependent wave propagation is demonstrated using the spectral method in the proximity of the saddle point.

2 FORMULATION OF MATHEMATICAL MODEL

In the present section, a mathematical model consisting of an infinitely extended floating elastic plate is developed under the assumption of small amplitude structural response and shallow water approximation. The interaction of gravity waves with the infinitely extended floating elastic plate/ice sheet generates the flexural gravity waves. Besides, a two-dimensional Cartesian coordinate system is assumed to represent the fluid domain with x-axis along the mean position of the plate covered surface and y-axis acting vertically downward. In addition, the fluid is assumed to be inviscid, incompressible and the flow is considered irrotational. Moreover, the plate is assumed to be thin, isotropic and homogeneous. Thus, the equation of continuity in the case of long waves yields

$$\frac{\partial \zeta}{\partial t} - h\frac{\partial^2 \Phi}{\partial x^2} = 0, \tag{1}$$

with $\Phi(x,t), \zeta(x,t)$ and h being the velocity potential, floating plate deflection and water depth respectively. The equation of motion associated with the long flexural gravity waves yields [5]

$$EI\frac{\partial^4 \zeta}{\partial x^4} + N\frac{\partial^2 \zeta}{\partial x^2} + \rho_p d\frac{\partial^2 \zeta}{\partial t^2} + \rho g \zeta = \rho\frac{\partial \Phi}{\partial t}. \tag{2}$$

where E is the Young's modulus, $I = d^3/12(1-\nu^2)$ with d and ν being the plate thickness and Poisson ratio respectively, N is the lateral compressive force, g is the acceleration due to gravity, ρ is the water density, ρ_p is the plate density and P_s is the pressure acting upon the floating elastic plate/ice sheet. Finally, combining Eqs. (1) and (2), the flexural gravity wave equation in shallow water is obtained as

$$D\frac{\partial^6 \zeta}{\partial x^6} + Q\frac{\partial^4 \zeta}{\partial x^4} + \gamma_p\frac{\partial^4 \zeta}{\partial x^2 \partial t^2} + \frac{\partial^2 \zeta}{\partial x^2} = \frac{1}{gh}\frac{\partial^2 \zeta}{\partial t^2}, \tag{3}$$

where $D = EI/\rho g$, $Q = N/\rho g$ and $\gamma_p = \rho_p d/\rho g$.

3 DERIVATION AND ANALYSIS OF DISPERSION RELATION

Assuming the motion is simple harmonic in time t with angular frequency ω, the plate deflection is written as

$$\zeta(x,t) = \mathrm{Re}\left\{\zeta_0 e^{\mathrm{i}(kx-\omega t)}\right\}, \tag{4}$$

with ζ_0 being the known amplitude. Substituting Eq. (4) in Eq. (3), the dispersion relation is obtained as

$$\omega^2 = \frac{gk^2h(Dk^4 - Qk^2 + 1)}{1 + gk^2h\gamma_p}. \tag{5}$$

It is worth mentioning that $\gamma_P \ll 1$; thus, the inertia term is neglected in the subsequent discussion as in [5]. Moreover, the phase and group velocities are derived in the form

$$c = \frac{\omega}{k} = \sqrt{gh(Dk^4 - Qk^2 + 1)}; \qquad c_g = \frac{d\omega}{dk} = \frac{gkh(3Dk^4 - 2Qk + 1)}{\omega}. \tag{6}$$

It is pertinent to mention that the phase velocity c vanishes for $Q = Q_c = 2\sqrt{D}$ m^2 and $k = k_c = (1/D)^{1/4}$ m^{-1}, whilst the group velocity c_g attains zero minimum for $Q = Q_{cg} = \sqrt{3D}$ m^2 and $k = k_{cg} = (1/3D)^{1/4}$ m^{-1}. Noticeably, both c_g and dc_g/dk vanish for Q_{cg} and k_{cg}. Further, Q_c and Q_{cg} are referred to as the buckling limit and threshold of blocking of compressive force respectively, whereas k_c and k_{cg} are known as the critical wavenumber and saddle point respectively [1]. may be noted that the vanishing of group velocity i.e., the existence of optima in the dispersion relation leading to the occurrences of flexural gravity wave blocking.
In Fig. 1, the dispersion curve is exhibited versus wavenumber for different values of compressive force. Figure 1 demonstrates that no wave blocking occurs for $Q < Q_{cg}$, whilst for $Q > Q_{cg}$ the dispersion curve possesses two optima which correspond with the primary (higher wavenumber) and secondary (lower wavenumber) blocking points. In addition, three propagating wave modes exist for any frequency within the limits of the blocking point.

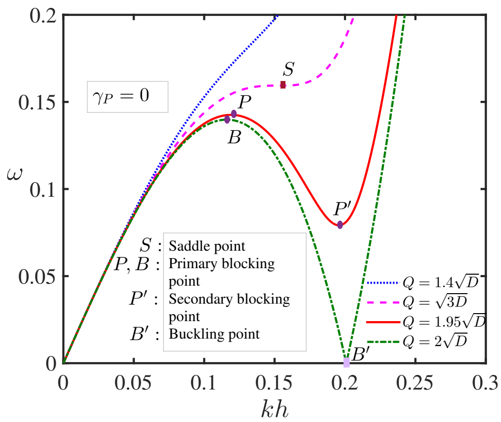


Figure 1: Dispersion curves for different values of Q with $\gamma_P = 0$, $EI = 5 \times 10^8$ Pa.

4 WAVE AMPLITUDE NEAR SADDLE AND BLOCKING POINTS

In the present section, emphasis is given for determining the wave amplitude in the vicinity of saddle as well as the blocking points for weakly varying compressive force $Q(x)$ in two different subsections.

4.1 WAVE AMPLITUDE NEAR SADDLE POINT

In the vicinity of saddle point, the wave amplitude $a(x,t)$ can be written as [3]

$$a(x,t) \equiv a(\varepsilon^{1/6}x, \varepsilon^{1/6}t), \tag{7}$$

with $\varepsilon \ll 1$ being a small parameter. Thus, the deflection of floating elastic plate $\zeta(x,t)$ is written in the form

$$\zeta(x,t) = a(\varepsilon^{1/6}x, \varepsilon^{1/6}t)e^{\mathrm{i}(kx-\omega t)}. \tag{8}$$

Next, we set $\overline{k}^4 = Dk^4$, $\overline{Q} = Q/\sqrt{D}$ and $\overline{\omega}^2 = (\sqrt{D}/gh)\omega^2$ and the overbar is omitted in the subsequent discussions. Consequently, substituting Eq. (8) in Eq. (3) and considering weakly varying compressive force $Q = Q(\varepsilon^{1/2}x)$, the Taylor series expansion of the compressive force Q about the saddle point, say $x = x_0$, yields

$$Q(\varepsilon^{1/2}x) = Q_0 + \varepsilon^{1/2}(x - x_0)Q'(x_0) + ..., \tag{9}$$

where $Q_0 = Q(x_0)$. Using Eq. (9) in Eq. (3) and comparing different powers of ε as in [3], it is obtained that

$$O(1) : \omega^2 = k^6 - Q_0k^4 + k^2, \tag{10}$$

$$O(\varepsilon^{1/6}) : \frac{\partial a}{\partial t} + c_g \frac{\partial a}{\partial x} = 0, \tag{11}$$

$$O(\varepsilon^{1/3}) : -\frac{\partial^2 a}{\partial t^2} + \left(15k^4 - 6Q_0k^2 + 1\right) \frac{\partial^2 a}{\partial x^2} = 0, \tag{12}$$

$$O(\varepsilon^{1/2}) : 4\mathrm{i}k(Q_0 - 5k^2)\frac{\partial^3 a}{\partial x^3} - \left\{(x - x_0)Q'(x_0)k^4\right\} a = 0, \tag{13}$$

$$\ldots$$

It may be noted that Eq. (10) is the dispersion relation of the long flexural gravity wave. On the other hand, the group velocity vanishes in the vicinity of the saddle point. Thus, Eq. (11) reveals that wave amplitude a is independent of time t in the proximity of saddle point. Again, at the saddle point both $c_g = 0$ and $dc_g/dk = 0$ [1], which ensures that Eq. (12) is trivially satisfied. Subsequently, Eq. (13) provides the solution near the saddle point. Consequently, $O(\varepsilon^m)$ for $m > 1/2$ is neglected. Setting $\xi = (x - x_0)\left\{\dfrac{Q'(x_0)k^3}{20k^2 - 4Q_0}\right\}^{1/4}$, Eq. (13) can be rewritten as

$$\frac{d^3 a}{d\xi^3} - \mathrm{i}\xi a = 0, \tag{14}$$

which is known as the hyper-Airy differential equation. Using the WKB theory [3], the asymptotic solution of Eq. (14) for $\xi \to -\infty$ is obtained as

$$a(\xi) \sim (-\xi)^{-1/3} e^{3e^{-\pi \mathrm{i}/6}\gamma(-\xi)^{4/3}/4}, \tag{15}$$

where γ is the cube root of unity. In the present study, the asymptotic solution is considered for $\gamma = -(1+\mathrm{i}\sqrt{3})/2$ due to the fact that it is decaying to the right and oscillatory in nature to the left about the point x_0. By solving Eq. (14) and using Eq. (15), the wave amplitude a is obtained in the form

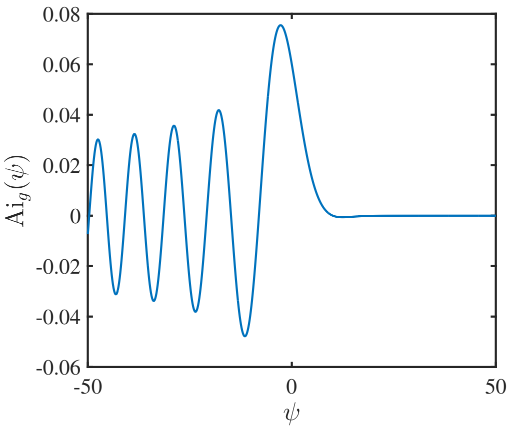


Figure 2: Generalised Airy function $\mathrm{Ai}_g(\psi)$

$$a \approx \frac{a_m}{(-\xi_m)^{-1/3} e^{3e^{-\pi \mathrm{i}/6}\gamma(-\xi_m)^{4/3}/4}} \mathrm{Ai}_g(\xi),$$

with

$$\mathrm{Ai}_g(\xi) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{\tau^4}{4} + \xi\tau\right) d\tau,$$

being the generalised Airy function, a_m being the amplitude at the point $x = x_m$, where $x_m = x_0 - O(\alpha)$ with α being very small which is similar to the thickness of the boundary layer as in [3].

4.2 WAVE AMPLITUDE NEAR BLOCKING POINTS

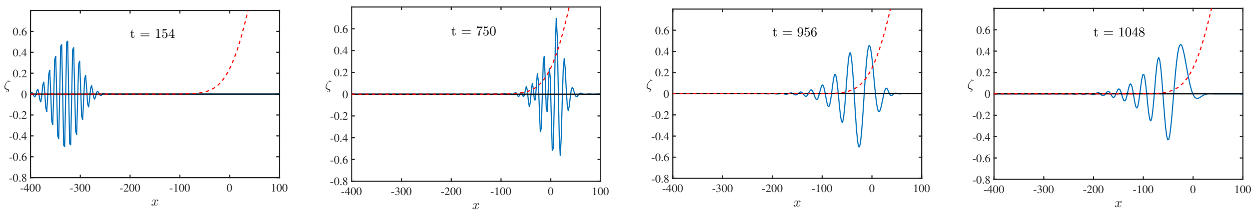
It may be recalled from section 3 that the group velocity vanishes in the close neighbourhood of primary as well as secondary blocking points. Thus, proceeding in a similar manner as in subsection 4.1 and considering $Q = Q(\varepsilon^{1/3}x)$, it can be easily derived that the wave amplitude near the blocking points satisfies

$$\frac{d^2a}{dx^2} - \frac{(x - x_0)Q'(x_0)k^4}{15k^4 - 6Q(x_0)k^2 + 1}a = 0, \tag{16}$$

which is the well-known Airy differential equation. The solution of Eq. (16) yields the amplitude $a(x)$ near the primary/secondary blocking point in terms of the Airy function $\mathrm{Ai}(x)$.

5 TIME-DOMAIN SIMULATION

In the present section, the time domain simulation of flexural gravity wave propagation in the vicinity of the saddle point is demonstrated. Figure 3 demonstrates that no wave transmission occurs beyond



(a) (b) (c) (d)

Figure 3: Gaussian pulse near the saddle point with slowly varying $Q(x)$ in red dashed line.

the saddle point ($x = 80$ m). Figure 3a reveals the Gaussian pulse for time $t = 154$ s. On the other hand, Fig. 3b depicts that the incoming wave amplitude attains a higher value in the proximity of the saddle point beyond which it decays, which is similar to the characteristics of generalized Airy function as in Fig. 2. Figures 3c and 3d exhibit that wave reflection occurs from the saddle point [3].

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