# A High-order Pseudospectral Incompressible Navier-Stokes Free Surface Model in 2D

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Simulation of water waves and taking into account the sea floor to estimate sea states are important for the design of offshore structures. We propose a new high-order accurate pseudo-spectral method for solving the incompressible Navier-Stokes equations with a free surface. The work is motivated by the lack of high-order accurate free surface water wave models which include viscous and rotational effects. The numerical scheme utilizes Fourier and Chebyshev basis functions for the numerical discretization and enables exploiting the fast Fourier transform (FFT) algorithm for efficiency reasons. A key feature is an explicit-implicit pressure-correction method designed to work with general Runge-Kutta (RK) methods used for temporal integration that fulfills mass balance and a pressure-velocity coupling. Another key feature is the use of a Fourier-continuation technique for solving numerical wave tank problems on finite (non-periodic) domains. As a starting point towards establishing the solver, we provide benchmark results to demonstrate accuracy and convergence using established cases used for potential flow solvers. We present numerical case results for i) nonlinear stream function wave solutions and ii) steep solitary wave reflection in a numerical wave tank in this abstract.

## GOVERNING EQUATIONS

We consider the general - potentially 3D - incompressible flow with constant properties governed by the Navier-Stokes equations on the following form:

$$\nabla \cdot \mathbf{v} = 0, \tag{1a}$$

$$\partial_t \mathbf{v} + \mathbf{v}\nabla \cdot \mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{v}, \tag{1b}$$

with $\mathbf{v}$ being the velocity field $[u, v, w]^T$, $\nabla$ the gradient operator $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}]^T$, $\rho$ the density, $p$ the pressure, $\mathbf{g}$ the gravitational acceleration $[0, 0, -g]^T$, and $\nu$ the kinematic viscosity. The equations describes conservation of mass and momentum, respectively, and both must be fulfilled to ensure accurate results. The total pressure $p$ is split into its static and dynamic parts $p = p_D + p_S$ with $p_S = \rho g(\eta - z)$, where $\eta(x, y, t)$ is the water free surface. Atmospheric pressure is ignored and the pressure on the free surface is assumed to be zero. The evolution of the free surface elevation is governed by the kinematic boundary condition, which ensures that free surface particles remain on the free surface

$$\frac{\partial \eta}{\partial t} + u|_{z=\eta}\frac{\partial \eta}{\partial x} + v|_{z=\eta}\frac{\partial \eta}{\partial y} = w|_{z=\eta}, \tag{2}$$

where $|_{z=\eta}$ denotes evaluation at the free surface level. The constantly evolving free surface elevation yields a time-dependent domain for which nonlinear waves requires re-discretization of the computational domain. To avoid this the Navier-Stokes equations are solved in a time-invariant $\sigma$-domain that is achieved by applying a modified version of the classical $\sigma$-transform

$$\sigma = \frac{2z + 2h(x, y) - d(x, y, t)}{d(x, y, t)}, \quad -1 \leq \sigma \leq 1, \tag{3}$$

where $h(x, y)$ is the elevation of the seabed and $d(x, y, t) = \eta(x, y, t)+h(x, y)$ is the total depth. The current $\sigma$-transform differs from the usual as it ranges from -1 to 1 due to the support of the Chebyshev basis The transformation limits the number of solvable problems as it does not allow for overturning or breaking waves since this leads to spatial singularities. Applying the chain rule, the Navier-Stokes equations is transformed to the $\sigma$-domain as,

$$\nabla_\sigma \cdot \mathbf{v} = 0, \tag{4a}$$

$$\partial_t \mathbf{v} + \mathbf{v}_\sigma \nabla \cdot \mathbf{v} = -\frac{1}{\rho}(\nabla_\sigma p_D + \nabla p_S) + \mathbf{g} + \nu \nabla_\sigma^2 \mathbf{v}, \tag{4b}$$

where the velocity in the $\sigma$-direction and the derivative operator is given as:

$$\mathbf{v}_\sigma = [u, v, w_\sigma]^T, \quad w_\sigma = \frac{\partial \sigma}{\partial t} + u\frac{\partial \sigma}{\partial x} + v\frac{\partial \sigma}{\partial y} + w\frac{\partial \sigma}{\partial z}, \tag{5a}$$

$$\nabla_\sigma = \left[\frac{\partial}{\partial x} + \frac{\partial \sigma}{\partial x}\frac{\partial}{\partial \sigma}, \frac{\partial}{\partial y} + \frac{\partial \sigma}{\partial y}\frac{\partial}{\partial \sigma}, \frac{\partial \sigma}{\partial z}\frac{\partial}{\partial \sigma}\right]^T. \tag{5b}$$

The $\sigma$-derivatives are evaluated from (3) by using the chain rule,

$$\begin{aligned} &\frac{\partial \sigma}{\partial t} = -d^{-1}\left(\frac{\partial d}{\partial t} + \sigma \frac{\partial d}{\partial t}\right), \quad \nabla \sigma = d^{-1}\left(2\nabla h - \nabla d - \sigma \nabla d\right), \\ &\nabla^2 \sigma = d^{-1}\left(2\nabla^2 h - \nabla^2 d - \sigma \nabla^2 d - 2\nabla \sigma \nabla d\right), \quad \frac{\partial \sigma}{\partial z} = 2d^{-1}, \end{aligned} \tag{6}$$

where the time derivative of the depth is evaluated at the free surface from the kinematic boundary condition (2) and projected down into the domain.

## NUMERICAL DISCRETIZATION

A method of lines approach is used to discretize the governing equation in time. The momentum equation (4b) and the free surface equation (2) are advanced in time by employing an explicit-implicit pressure-correction method using a general RK method from [1] (here for velocity)

$$\mathbf{v}^{(k)} = \sum_{l=1}^{k} \alpha_{kl}\mathbf{v}^{(l-1)} + \beta_{kl}\Delta t f(\mathbf{v}^{(l-1)}), \quad k = 1, 2, ..., s, \tag{7}$$

with $\mathbf{v}^{(0)} = \mathbf{v}^n$, $\mathbf{v}^{(s)} = \mathbf{v}^{n+1}$ and $f(\cdot)$ being the $\sigma$-transformed momentum (4b). Conservation of mass is ensured at each time step by applying a pressure-correction method originally proposed by [2], where the pressure is determined such that the velocity field is divergence free. Continuity needs to be satisfied at each RK-stage, hence the divergence $\nabla_\sigma^{(k)} \cdot \mathbf{v}^{(k)}$ at stage $k$ must vanish. Note that the derivative operator is also denoted with the stage index as it is time-dependent on the discrete level due to the $\sigma$-transform. Let $\hat{f}(\cdot)$ be the momentum equation without the dynamic pressure and the divergence at stage $k$ becomes

$$\nabla_\sigma^{(k)} \cdot \mathbf{v}^{(k)} = \nabla_\sigma^{(k)} \cdot \sum_{l=1}^{k} \alpha_{kl}\mathbf{v}^{(l-1)} + \beta_{kl}\Delta t \left[\hat{f}(\mathbf{v}^{(l-1)}) - \frac{1}{\rho}\nabla_\sigma^{(l-1)} p_D^{(l-1)}\right]. \tag{8}$$

We set $\nabla_\sigma^{(k)} \cdot \mathbf{v}^{(k)} = 0$ and by isolating the pressure, the velocity field at stage $k$ becomes divergence free by solving the Poisson type problem

$$\nabla_\sigma^{(k)} \cdot \nabla_\sigma^{(k-1)} p_D^{(k-1)} = \frac{\rho}{\beta_{kk}\Delta t}\nabla_\sigma^{(k)} \cdot \hat{\mathbf{v}}^{(k)}, \tag{9}$$

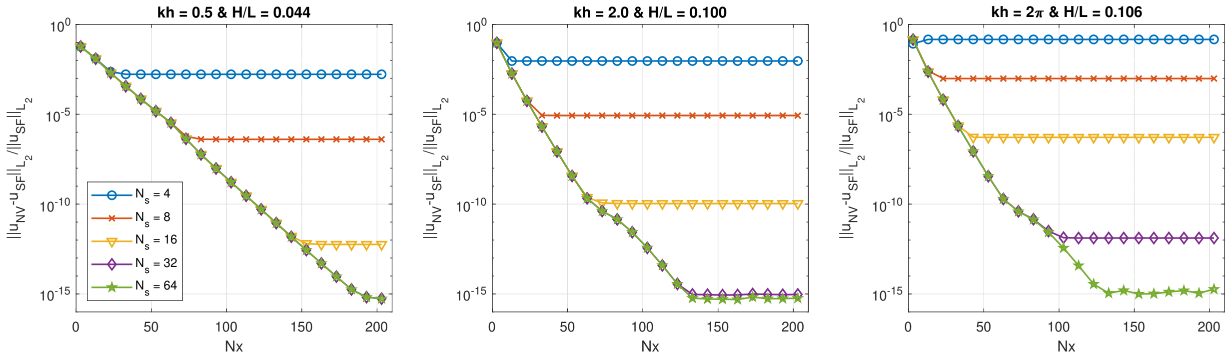


Figure 1: Relative $L_2$-error of the velocity $u$ measured against a stream function solution for kh = 0.5, 2.0 and $2\pi$ with maximum theoretical steepness of 75%.

where the velocity without the pressure $\hat{\mathbf{v}}^{(k)}$ is given as:

$$\hat{\mathbf{v}}^{(k)} = \sum_{l=1}^{k-1} \alpha_{kl}\mathbf{v}^{(l-1)} + \beta_{kl}\Delta t f(\mathbf{v}^{(l-1)}) + \alpha_{kk}\mathbf{v}^{(k-1)} + \beta_{kk}\Delta t \hat{f}(\mathbf{v}^{(k-1)}). \tag{10}$$

The boundary value problem is solved with a zero Dirichlet pressure boundary condition at the free surface and a no-slip Neumann condition at wall boundaries, where the latter is determined in a similar fashion as the Poisson problem resulting in:

$$\mathbf{n} \cdot \nabla_\sigma^{(k-1)} p_D^{(k-1)} = \frac{\rho}{\beta_{kk}\Delta t}\mathbf{n} \cdot \hat{\mathbf{v}}^{(k)}, \tag{11}$$

with $\mathbf{n}$ being the normal vector. The velocity without pressure (10) can now be corrected with the newly obtained pressure to determine the final divergence free velocity. The Poisson type equation consists of mixed-stage derivative operators and it is necessary to advance the free surface to stage $k$ before the Poisson equation is set up. The presented results uses the classical explicit fourth-order RK-scheme.

The governing equations are discretized in a 2D $\sigma$-domain using a pseudospectral method based on Fourier modes in the horizontal direction evaluated at $N_x$ equidistant nodes and Chebyshev polynomials in the vertical $\sigma$-direction evaluated at $N_s$+1 Chebyshev-Gauss-Lobatto nodes. The Fourier basis is inherently periodic, hence enforcing boundary conditions are only necessary at the free surface and the seabed. A collocation approach is used to represent the free surface, the velocities, and the pressure, which are discretized in nodal space by global Lagrange interpolation polynomials $l_j(x_i)$ with the Cardinal property that $l_j(x_i) = \delta_{ij}$, where $\delta_{ij}$ is Kroneckers delta. The derivatives in the horizontal direction are evaluated discreetly in modal space by FFT and the vertical derivatives are evaluated by fast Chebyshev transform (FCT). Mixed derivatives are evaluated by repeated use of the FFT and FCT. These discrete evaluations makes the computations of the derivatives scalable as they both have work effort $\mathcal{O}(N \log N)$. Nonlinear terms are evaluated at collocation points by direct products, that are susceptible to aliasing, which can be damped through mild spectral filtering.

## NONLINEAR ACCURACY

The accuracy of the nonlinear model is shown in Fig. 1 where the relative $L_2$-error of the bulk velocity $u$ is shown. The error is computed for a 2D nonlinear stream function wave with limiting steepness of 75% and dimensionless depth $kh$ = 0.5, 2.0, and $2\pi$. The test shows spectral convergence and that more horizontal resolution is needed on shallow water and more

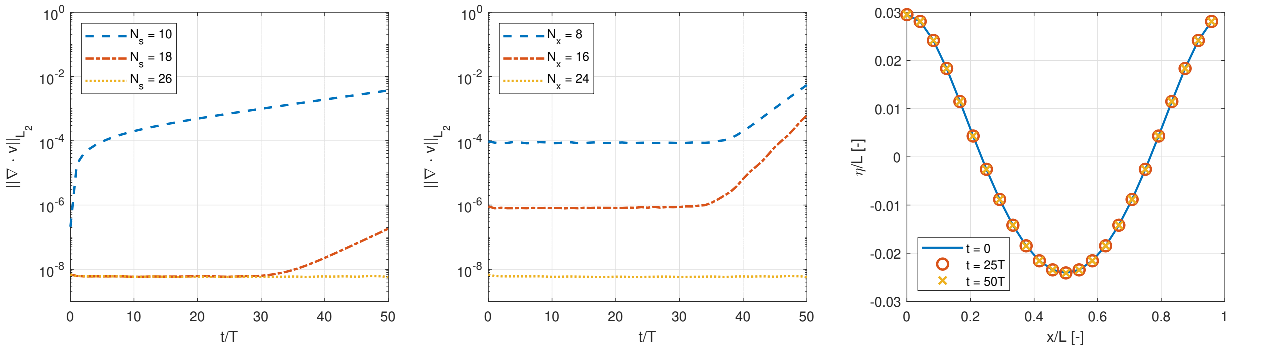


Figure 2: Stream function wave for kh 2.0 with maximum theoretical steepness of 40% and $\Delta$t = T/100. Divergence for different resolution with constant $N_x = 24$ (left) and constant $N_s$=26 (middle), and a comparison of the free surface elevation for $N_x = 24$ and $N_s$=26. (right)

vertical resolution is needed on deep water, which is as expected due to the decay of modal coefficients for the stream function.

Stability of the model is highly dependent on the divergence which can be seen in Fig. 2 (left and middle) for a 2D nonlinear stream function wave propagating for 50 wave periods with limiting steepness of 40% and $kh = 2.0$. The test hints that the vertical resolution is of great importance for the stability but the magnitude of the divergence is governed by the horizontal resolution. Fig. 2 (right) show the free surface elevation for the stable wave propagation test where excellent agreement with the constant low divergence can be observed.

The model's ability to cope with nonlinear shallow-water waves is tested by colliding two identical 2D solitary waves, essentially replicating a solitary wave reflecting off a vertical wall. The free surface elevation of the solitary wave at attachment ($\eta_a$), maximum wave run up ($\eta_0$), and detachment ($\eta_d$) from the wall are shown in Fig. 3 with a comparison to results from [3]. The maximum run up is determined up to $a/h = 0.6$ as it was found by [4] that a jet occurs for larger $a/h$, hence breaking the invertibility of the $\sigma$-transform.

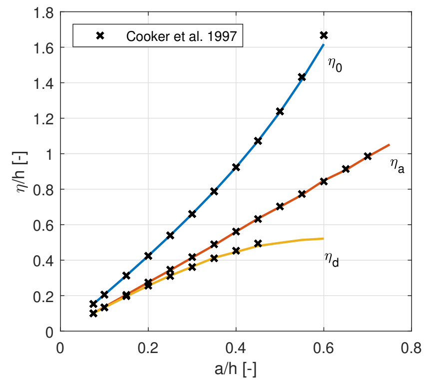


Figure 3: Attachment ($\eta_a$), maximum wave run up ($\eta_0$) and detachment ($\eta_d$) curves for reflection of a solitary wave.

At the workshop we plan to show i) a spectral filter for dealiasing of nonlinear simulations for improving model robustness and ii) an extension of the wave tank used for the reflecting solitary wave with a benchmark comparison to experimental results of a submerged bar test [5].

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